

Fundamentals of Probability

Introduction

Probability is the likelihood that an event will occur under a set of given conditions. The probability of an event occurring has a value between 0 and 1. An “impossible event” would have a probability of 0; a “certain event” would have a probability of 1.

$$0 \leq P(A) \leq 1.$$

This packet will focus on the foundation of probability and introduce different techniques for solving probability questions. When answering probability questions, use an exact fraction or round to 3 significant digits (leading zeros are not significant) unless told otherwise.

Classical (Theoretical) Probability Formula

For the Classical Probability Formula, the outcomes must be equally likely. If the outcomes are not equally likely, then the Empirical Probability Formula should be used.

$$P(A) = \frac{s}{n} = \frac{\text{(Number of ways event A can occur)}}{\text{(Total number of all outcomes)}}$$

Example: What is the probability of drawing a 7 from a standard deck of 52 cards?

Solution: In a standard deck of cards, there are 4 suits of 13 cards each: Spades (black), Clubs (black), Hearts (red), and Diamonds (red). Each suit contains the following: the numbers 2 through 10, Jack, Queen, King, and Ace. Therefore, a standard deck of cards contains four 7s.

$$P(\text{drawing a 7}) = \frac{4}{52} = \frac{1}{13} = .0769$$

Empirical Probability Formula (Actual data)

$$P(A) = \frac{\text{(Number of times event A occurred)}}{\text{(Number of times experiment performed)}}$$

Note: The more times an experiment is repeated, the closer the Empirical probability will come to the theoretical probability.

The question does not always provide the total number of outcomes. If the problem does not specify the total number of outcomes, there are different ways of finding this quantity depending on the situation.

Finding Total Number of Outcomes

When the number of choices is small, the creation of a table or other visual diagram can assist in determining the number of outcomes.

Example: What is the probability of a couple having exactly 3 girls in a family of 4? Assume the probability of having a girl is equally as likely as having a boy, and the gender of one child does not influence the gender of the others.

Solution: The first step is to determine the total number of possible outcomes of children in a family of four. Since there are only two choices, boy or girl, we can find the number of total outcomes by evaluating 2^n , where the n is the number of children in the family.

$$2^4 = 16$$

Since the number is small, it is easy to make a chart of all the different combinations of four children. All the possible gender combinations a couple could have with four children are listed within the table to the right.

1 st Child	2 nd Child	3 rd Child	4 th Child
Boy	Boy	Boy	Boy
Boy	Boy	Boy	Girl
Boy	Boy	Girl	Boy
Boy	Boy	Girl	Girl
Boy	Girl	Boy	Boy
Boy	Girl	Boy	Girl
Boy	Girl	Girl	Boy
Boy	Girl	Girl	Girl
Girl	Boy	Boy	Boy
Girl	Boy	Boy	Girl
Girl	Boy	Girl	Boy
Girl	Boy	Girl	Girl
Girl	Girl	Boy	Boy
Girl	Girl	Boy	Girl
Girl	Girl	Girl	Boy
Girl	Girl	Girl	Girl

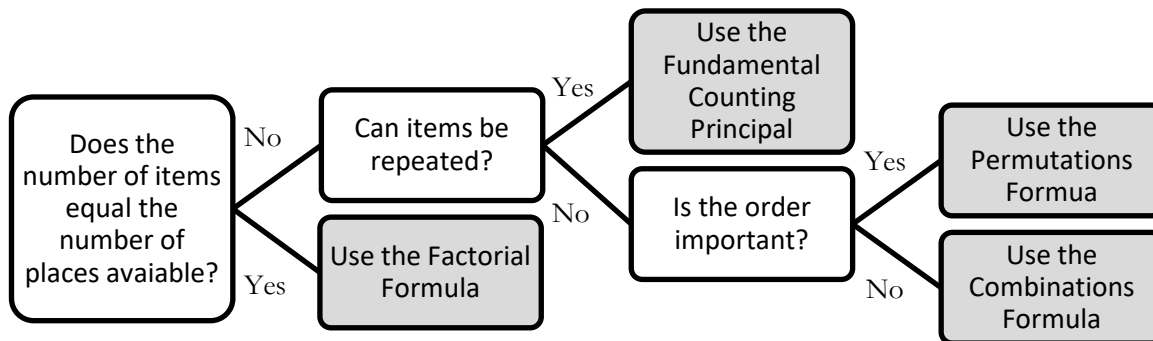
There are 16 different gender combinations a couple can have with four children. Of those 16 ways, only 4 combinations result in exactly 3 girls. So the probability of a family having exactly three girls is:

$$P(3 \text{ girls in 4 births}) = \frac{4}{16} = \frac{1}{4} = .25$$

Sometimes it is not practical to use a table or other listing methods to find total number of outcomes. The following methods are often used to calculate the total number of outcomes.

Counting Methods

The diagram below can aid in determining which Counting Method is most appropriate for a question.



Factorial Formula

When arranging all of the items in the same number of places, use the Factorial Formula method.

Example: Find the total number of ways 7 people can sit in 7 empty seats at a movie theater.

Solution: To find the total number of ways 7 people can sit in 7 empty seats at a movie theater, consider filling one seat at a time. The first seat would be open to all 7 people. Once someone sits down, there are only 6 people left to sit in the next open seat. Then the next person sits down, and there are only 5 people left to sit in the next open seat. This process continues until there is only one person and one seat left. Mathematically, this example can be expressed as “7!” (7 factorial) and is defined as:

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

The total number of ways 7 people can sit in the 7 empty movie theater seats is: **5,040**.
Remember: $0! = 1$.

You can use your TI 83/84 Plus to calculate the Factorial Formula:

1. First input value for total number of objects.
2. Press **MATH**
3. Use the **Arrow Keys** to highlight **PRB**
4. Use **Arrow Keys** to highlight **4: !**
5. Press **ENTER**

Fundamental Counting Principal (FCP)

When repetitions are allowed and the number of ways to fill an open place is not affected by the way in which previous places are filled, the Fundamental Counting Principal method should be used. Multiply the number of ways each part of the task can be accomplished together.

Example: A Virginia license plate consists of 3 letters followed by 4 numbers. If repeated letters and numbers are allowed, how many different license plates can be created? What is the probability of a license plate containing only one “A?”

Solution: To find the total number of possible standard license plate options for the state of Virginia, first define the number of spaces on each plate. The standard plate consists of 3 letters followed by 4 numbers. Since there are 26 letters in the alphabet, there are 26 options for the first 3 spaces. The 4 number spots must be filled with a single digit 0-9, so there are 10 options for the next 4 spaces.

26	26	26	10	10	10	10
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The total number of outcomes is:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$$

Now, to find the number of license plates that contain only one “A,” look at each space again. If the “A” were to appear in the first letter space, there is only one letter choice for the first letter space. Now consider the next letter space. Since the problem specifies only one “A” on the license plate, the letter “A” is not repeated. Therefore, only 25 letters are available to choose from for the second and third letter spaces; the number spaces are unaffected.

1	25	25	10	10	10	10
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The total number of license plates with only one “A” in the first space is:

$$1 \times 25 \times 25 \times 10 \times 10 \times 10 \times 10 = 6,250,000$$

If you evaluate the one “A” located in the second or third space, you would get the same number of license plates as if the “A” were located in the first space. The total number of ways to have one “A” on a license plate is:

$$6,250,000 + 6,250,000 + 6,250,000 = 18,750,000.$$

Therefore, the probability of having one “A” on a license plate is:

$$\frac{18,750,000}{175,760,000} = 0.107$$

Permutations

When replacements are not allowed, and the order in which the items rank is important, use the Permutations method. The formula for calculating permutations is:

$${}_n P_r = \frac{n!}{(n-r)!}$$

The “n” is the total number of items, and the “r” is the number of ways they are being selected.

Example: The student body is electing the class President, Vice-President, Treasurer, and Secretary from a group of 14 students consisting of 5 seniors, 4 juniors, 3 sophomores, and 2 freshmen. What is the probability of electing all seniors?

Solution: There are at least two ways to solve this problem. This solution will only cover the Permutations method. To begin we need to know how many different ways we can elect the class representatives. Since order matters to this problem, and the same person cannot be elected to two positions, use permutations.

$${}_{14} P_4 = \frac{14!}{(14-4)!} = 24,024$$

Next we want to determine how many of the 24,024 arrangements consist of only seniors. Therefore, we want to determine how many ways we can arrange only the 5 seniors as the class representatives.

$${}_5P_4 = \frac{5!}{(5-4)!} = 120$$

The probability of electing all seniors as the class representatives is:

$$\frac{120}{24,024} = 0.005$$

In the following example, two different techniques will be needed to find the probability.

Example: The student body is re-electing the class President, Vice-President, Treasurer, and Secretary from a group of 14 students consisting of 5 seniors, 4 juniors, 3 sophomores, and 2 freshmen. This time each class must be represented. What is the probability of electing one officer from each class?

Solution: For this problem, analyze how many ways a member of each class can be elected and then how many ways we can arrange the four elected members in the positions similar to the Fundamental Counting Principal. How many ways can one member of each class be elected?

The number ways a senior could be elected is: ${}_5P_1 = 5$

The number ways a junior could be elected is: ${}_4P_1 = 4$

The number ways a sophomore could be elected is: ${}_3P_1 = 3$

The number ways a freshman could be elected is: ${}_2P_1 = 2$

How many different ways can the four elected class members be arranged in the four offices?

$${}_4P_4 = 24$$

How many total ways can the class representatives be elected?

$${}_{14}P_4 = \frac{14!}{(14-4)!} = 24,024$$

Put all the information together to form the probability. The numerator will represent the number of ways one member of each class can be elected and arranged in the four offices. The denominator will represent the total number of ways the officers can be elected.

$$\frac{{}_5P_1 \times {}_4P_1 \times {}_3P_1 \times {}_2P_1 \times {}_4P_4}{{}_{14}P_4} = \frac{5 \times 4 \times 3 \times 2 \times 24}{24,024} = 0.1199$$

Use the TI 83/84 Plus to calculate Permutations:

1. Input value for “n”

2. Press **MATH**
3. Use the **Arrow Keys** to highlight **PRB**
4. Use the **Arrow Keys** to highlight **2: nPr**
5. Press **ENTER**
6. Input value for “r”
7. Press **ENTER**

Combinations

When replacements are not allowed and the order in which items are ranked does not matter, use the Combinations method. The formula for combinations is:

$${}_n C_r = \frac{n!}{(n-r)!r!}$$

Example: In order to win the Pick 5 lottery, a participant must select the correct 5 numbers in any order from between 1 and 52. What is the probability of winning the Pick 5 lottery?

Solution: Since the order of the numbers does not matter, we can use combinations to determine the total possible number of combinations of 5 numbers from 1-52.

$${}_{52} C_5 = \frac{52!}{(52-5)!5!} = 2,598,960$$

Since there is only one way to select the correct 5 numbers in any order, the probability of winning the Pick 5 lottery is:

$$\frac{1}{2598960} = 0.000000385$$

Use the TI 83/84 Plus to calculate Combinations:

1. Input value for “n”
2. Press **MATH**
3. Use the **Arrow Keys** to highlight **PRB**
4. Use the **Arrow Keys** to highlight **3: nCr**
5. Press **ENTER**
6. Input value for “r”
7. Press **ENTER**

Addition Rule

A compound event combines two simple events. $P(A \text{ or } B) = P(\text{event } A \text{ occurs or event } B \text{ occurs or they both occur})$. The formal addition rule is written as:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Notice that the quantity $P(A \text{ and } B)$ is subtracted. This is because the events that fall under both event A and event B at the same time are counted twice: once when calculating $P(A)$, and then

again when calculating $P(B)$. If event A and event B are mutually exclusive, or cannot happen at the same time, the formula simplifies to:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example: What is the probability of selecting a test subject who received a negative test result or lied? Use the chart below.

	Subject did not lie	Subject lied
Positive Test Result (Test indicated subject lied)	13	74
Negative Test Result (Test indicated subject did not lie)	68	6

Solution: The total number of people tested is:

$$13 + 74 + 68 + 6 = 161.$$

To determine the number of successes, identify the two events in the probability. The two events are: subjects who lied and subjects who had negative test results. The number of subjects who lied is:

$$74 + 6 = 80.$$

The number of subjects who received a negative test results is:

$$68 + 6 = 74.$$

Notice the 6 is counted in both individual probabilities. Now apply the addition rule to get:

$$80 + 74 - 6 = 148.$$

$$P(\text{negative result or lied}) = \frac{148}{161} = 0.919$$

The Complement

The complement of an event, A, consists of all outcomes where event A does not occur $P(\text{not } A)$ or $P(\bar{A})$. The sum of the probability of an event occurring and the probability of the event not occurring is equal to 1. This property is called the Rule of Complementary Events.

$$P(A) + P(\bar{A}) = 1 \quad \text{or} \quad P(A) = 1 - P(\bar{A}) \quad \text{or} \quad P(\bar{A}) = 1 - P(A)$$

The probability of event A not occurring is used in calculating the odds. Odds are always written in the most reduced terms.

Odds in favor: favorable : unfavorable $P(A) : P(\bar{A})$
 Odds against: unfavorable : favorable $P(\bar{A}) : P(A)$

Example: The Weather Channel predicted a 35% chance of rain today. What are the odds against rain?

Solution: The probability of rain is 35%; the probability of no rain is 65%. The odds against rain are 65:35. Both 65 and 35 can be divided by 5. The final result is: 13:7.

The complement is also useful in solving several probability questions involving the phrase “at least.” The phrase “at least one” means one or more. Calculating the probability of an event occurring one time, two times, three times, four times, or n times can be long and involved. In these instances, calculating the complement can be faster. The complement of “at least one” is zero or none.

$$P(\text{at least one}) = 1 - P(\text{none})$$

Example: If a classroom contains 5 students, what is the probability that at least two of them have the same birthday?

Solution: We start by determining the probability that none of the students have the same birthday. The probability of the first student having a birthday on any day is $\frac{365}{365}$. The probability of the next student not having the same birthday as the first student is $\frac{364}{365}$. The probability of the third student not having the same birthday as the first two students is $\frac{363}{365}$, and so on until we have all 5 students.

$$P(\text{none the same}) = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \frac{361}{365} = 0.973$$

The probability of at least two students having the same birthday is:

$$P(\text{at least two birthdays the same}) = 1 - 0.973 = 0.027$$

Conditional Probability

The conditional probability is the probability of event B occurring after it is assumed that event A has already occurred. The conditional probability is denoted: $P(B|A)$. Conditional probability is used with $P(A \text{ and } B)$ when the “and” indicates the probability of event A occurring followed by event B, not when “and” indicates the probability of event A and B happening at the same time. Conditional probability occurs without replacement. There are two types of conditional probabilities: dependent and independent.

Events are considered dependent when the outcome of the first event affects the outcome of the second event. For example: drawing a heart from a standard 52 card deck followed by a King.

Selecting a heart from the deck of cards and selecting a King from the deck of cards are dependent because if the King of Hearts is drawn first, there are now only three Kings left in the deck.

For dependent events, the following conditional formula is used*:

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

*If a sample is no more than 5% of the population, it should be treated as independent even though it is technically dependent.

Events are considered independent when the outcome of the first event does not affect the outcome of the second event. For example: drawing a spade from a standard 52 card deck followed by a diamond. When a Spade is removed from the deck, it does not affect the number of Diamonds remaining in the deck. In this case, the conditional formula above simplifies to:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 1: What is the probability of drawing a Queen, a three, and then either a Queen or a three in that specific order? Assume there are no replacements.

Solution 1: In a standard deck of cards, there are 4 suits of 13 cards each: Spades (black), Clubs (black), Hearts (red), and Diamonds (red). Each suit contains the following: the numbers 2 through 10, Jack, Queen, King, and Ace.

The probability of drawing a Queen for the first card is:

$$P(Q) = \frac{4}{52}$$

Drawing a Queen and drawing a three are independent events since taking a Queen out of the deck does not affect the number of threes remaining in the deck. The denominator is 51 because the first Queen drawn is not replaced. The probability of the second card being a three is:

$$P(3) = \frac{4}{51}$$

The probability of the third card being a Queen or a three is affected by the first two cards drawn and is therefore dependent. To calculate this probability, assume the first card drawn was a Queen, and the second card drawn was a three. How many Queens and threes are left in the deck? There are only 3 Queens and 3 threes. The probability of the third card being a Queen or a three is:

$$P(Q \text{ or } 3|Q|3) = \frac{3+3}{50} = \frac{6}{50}$$

Finally, multiply the probabilities together to find the probability.

$$P(Q, 3, Q \text{ or } 3) = \frac{4}{52} \times \frac{4}{51} \times \frac{6}{50} = 0.000724$$

Example 2: The local tire company produced a batch of 100 tires. Four of the tires are defective. What is the probability a customer randomly selects three good tires?

Solution 2: First, determine the number of tires in the batch that are good.

$$100 - 4 = 96$$

Since the number of tires to choose from changes every time a tire is selected, the events are dependent. Check the 5% guideline to see if the events can be treated as independent even though they are technically dependent.

$$100 \times 0.05 = 5$$

Since three tires are less than five tires, the boundary of the 5% guideline, the events can be treated as independent even though they are technically dependent.

$$\frac{96}{100} \times \frac{96}{100} \times \frac{96}{100} = \left(\frac{96}{100}\right)^3 = 0.8847$$

This will be very close to the answer calculated by treating the events as dependent.

$$\frac{96}{100} \times \frac{95}{99} \times \frac{94}{98} = 0.8836$$

Practice Problems

1. A three digit number is created by selecting three random digits from the following set:

$$\{0,1,2,3,4,5\}$$

Numbers beginning with 0 are not permitted. What is the probability that one randomly selected number generated from the above list is not a multiple of 5?

2. Three men and three women are waiting to interview for a job. If the candidates are called in for their interviews in a random order, what is the probability that all three women will be interviewed first?
3. A student does not want to be late for his final exam in Statistics, so he sets the alarm on three battery-operated alarm clocks. If each individual alarm clock has a 10% chance of failing, what is the probability of all three alarm clocks failing?
4. What is the probability of drawing three random cards from a standard 52-card deck in the following order: a Jack, a Spade, a red non-face card (2, 3, 4, 5, 6, 7, 8, 9, or 10)?
5. “Five-card Draw” is a game using a standard 52-card deck where the players each draw a hand of five cards. A royal flush, consisting of the Ace, King, Queen, Jack, and ten from the same suit, is the best hand a player can have. What is the probability that a player will randomly draw a royal flush?
6. If three people are selected at random from the blood type chart below, what is the probability that at least one of them will be type O?

Blood Type Chart				
	A	B	AB	O
Rh+	58	59	32	16
Rh-	18	10	5	2

7. What is the probability that a randomly selected person from the chart above will have blood type B given that his/her blood is Rh+?
8. Joanna, a Statistics student, sits down to take her final exam and realizes that she doesn't know the answers to any of the questions because she didn't study at all. If there are 10 multiple choice questions with the options of A through D for each, and on the grading scale a 70% is required for a passing C, what is the probability that Joanna will get a 70% on her test if she randomly bubbles in answers? Assume that all questions are of equal point value and there is an equal likelihood that any letter will be correct.

Solutions:

1. $\frac{2}{3}$

2. $\frac{1}{20}$

3. 0.001

4. 0.007

5. 0.00000154

6. 0.2476

7. 0.358

8. 0.0000061