

## Permutations and Combinations

In statistics, there are two ways to count or group items. For both permutations and combinations, there are certain requirements that must be met: there can be no repetitions (see permutation exceptions if there are), and once the item is used, it cannot be replaced. Both counting methods have  $n$  different items available, taken  $r$  at a time. The distinguishing aspects of the two different types of counting methods are as follows:

Permutations	Combinations
The order of the items is important.	The items do not have to be in any particular order.
Rearrangements of the same items are in different sequences.	Rearrangements of the same items are in the same sequence.
Use equation ${}_nP_r = \frac{n!}{(n-r)!}$	Use equation ${}_nC_r = \frac{n!}{r!(n-r)!}$

### Permutation: When All Items are Different

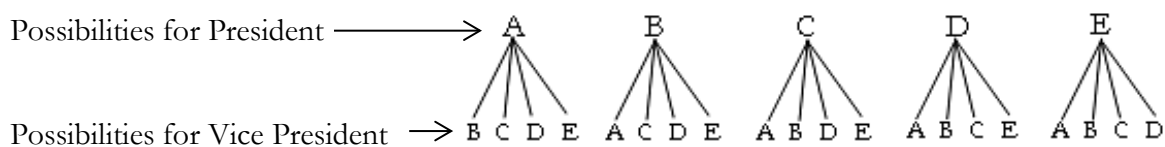
**Example:** There are students A, B, C, D, and E. How many ways can a President and a Vice President be selected from this group?

This example is classified as a permutation because the order in which students are selected is important due to the fact that the ranks of President and Vice President are not the same. If student A is chosen as President and student B is chosen as Vice President, the outcome is different than if student A is chosen as Vice President and student B is chosen as President.

**Method 1:** List all possibilities.

This method is simple when numbers are small, but they might not be appropriate for more complex problems. It may be prudent to do a diagram in order to have a visual display of all possibilities.

List all of the possibilities for the first option, in this case candidates for President. From there, branch out and list the possibilities for the next option, in this case Vice President, and so on until all required options are listed.



Count the outermost set of options, which in this case is the bottom row. There are 20 different ways to select a President and Vice President in this example.

**Method 2:** Multiply.

With this method, decide which event to determine first and then figure out the possibilities for that event. For this example assume that the President is elected first. Because there are 5 candidates running for the position, there are 5 possible outcomes for this event. Now move on to the next event, electing a Vice President. Because one of the students is already elected President, only 4 remain as possibilities for Vice President.

$$\underline{5} \times \underline{4} = 20$$

Multiply the possibilities for each event together. In this example there are 20 different ways to select a President and a Vice President from this group.

**Method 3:** Use the equation for permutations.

Use the equation  ${}_nP_r = \frac{n!}{(n-r)!}$  where  $P$  stands for permutation,  $n$  is the number of objects, and  $r$  is how many objects you are taking at a time. This method may be used in every situation except where there are restrictions or stipulations.

There are 5 students running, so the  $n$  value is 5. This example calls for a group of 2 students to be elected, one President and one Vice President, so the  $r$  value is 2.

$${}_5P_2 = \frac{5!}{(5-2)!}$$

Simplify the denominator.

$${}_5P_2 = \frac{5!}{3!}$$

The exclamation points represent factorials. Factorials are the product of the specified number and all of the numbers that came before it. In the numerator,  $5!$  is equivalent to 120. In the denominator,  $3!$  is equivalent to 6. Remember that  $0!$  always equals 1, not 0. To calculate factorials on the TI-83/84, first select the number in question and then press the *Math* button, arrow over to *PRB*, and select option four.

$${}_5P_2 = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{120}{6}$$

Simplify the fraction.

$${}_5P_2 = 20$$

Using the permutation equation, this group of students can be elected in 20 possible ways.

Use TI 83/84 to calculate Permutations:

1. First input value for n.
2. Press MATH and select PRB, then choose option 2:nPr and press the ENTER key.
3. Input value for r and press the ENTER key.

### Fundamental Counting Principle

Some circumstances involve restrictions on certain numbers. The fact that there are restrictions means that instead of considering all possibilities, only specific cases are allowed. In these cases it is best to use the multiplication method.

**Example:** How many three-digit area codes can be created using the numbers 0-9 if the area code cannot begin with a zero or a one, there must be a zero or a one as the second digit, and it cannot end in two zeros or two ones?

As with the multiplication method explained above, first decide on the first event. In this example, simply start with the first digit of the area code. The restriction above says that the area code cannot begin with a 0 or a 1. That leaves 8 other possibilities: 2, 3, 4, 5, 6, 7, 8, and 9.

$$\underline{8} \times \underline{\quad} \times \underline{\quad}$$

The restriction on the second digit is that the next number must be 0 or 1. This leaves us with only those 2 possibilities.

$$\underline{8} \times \underline{2} \times \underline{\quad}$$

The last restriction is that the second and third digit cannot be both 0 or both 1. If the second digit is 0, the possibilities for the third digit are 1, 2, 3, 4, 5, 6, 7, 8, and 9. If the second digit is 1, the possibilities for the third digit are 0, 2, 3, 4, 5, 6, 7, 8, and 9. In either situation, there are 9 possibilities for the last area code digit.

$$\underline{8} \times \underline{2} \times \underline{9} = 144$$

Multiply the possibilities for each digit together. These restrictions yield 144 area code combinations.

**Permutation Exceptions: Where Some Items are Identical**

**Situation 1:** Use the equation  $P = \frac{n!}{n_1!n_2!\dots n_k!}$  where  $P$  stands for permutation,  $n$  is the total number of objects,  $n_1$  is the number of a type of identical objects,  $n_2$  is the number of a type of identical objects, et cetera for all of the identical groups.

**Example:** A family goes to a studio to have their photo taken, and the photographer must arrange them to get the best shot. Three of them have red shirts, four of them have blue shirts, and two of them have black shirts. How many ways can the photographer arrange the family members?

In this example there are 9 family members total, regardless of shirt color, so the  $n$  value is 9. In the denominator the 3 family members wearing red would be the  $n_1$  value, the 4 family members wearing blue would be the  $n_2$  value, and the 2 family members wearing black shirts would be the  $n_3$  value.

$$P = \frac{9!}{3! 4! 2!}$$

In the numerator, 9 factorial is 362,880 in the numerator, and 3!, 4!, and 5! multiplied together in the denominator equal 288.

$$P = \frac{9!}{3! 4! 2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = \frac{362880}{288}$$

Simplify the fraction by dividing.

$$P = 1260$$

There are 1,260 ways to arrange the nine family members by shirt color.

**Situation 2:** If there are only two different types of items that are identical, use the equation  $P = \frac{n!}{(n-x)!x!}$  where  $P$  stands for permutation,  $n$  is the total number of items, and  $x$  is the number of items that are identical from one of the groups (it does not matter which group you choose).

**Example:** A family goes to a studio to have their photo taken, and the photographer must arrange them to get the best shot. Three of them are male and five of them are female. How many ways can the photographer arrange the family members?

In this example there are a total of 8 family members regardless of gender, so the  $n$  value is 8. Either group of identical objects can be substituted as the  $x$  value because the denominator comes out the same no matter what. Notice that, whatever the case, the sum of the numbers in the denominator (5 and 3) is always equal to the number in the numerator (8).

If men are the  $x$  value:

$$P = \frac{8!}{(8-3)!3!}$$

$$P = \frac{8!}{5!3!}$$

If women are the  $x$  value:

$$P = \frac{8!}{(8-5)!5!}$$

$$P = \frac{8!}{3!5!}$$

In the numerator,  $8!$  equals 40,320 while in the denominator  $3!$  multiplied by  $5!$  equals 720.

$$P = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{40320}{720}$$

Simplify the fraction.

$$P = 56$$

There are 56 ways the photographer can arrange 3 men and 5 women.

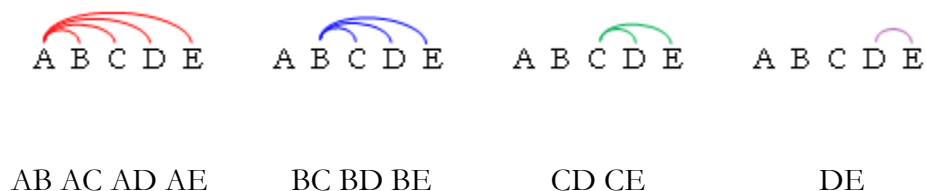
### Combination

**Example:** There are students A, B, C, D, and E. How many ways can two students from this group be chosen for a committee?

This example is classified as a combination because it does not matter which order the students are selected because both positions are equal; the first committee chair is the same as the second committee chair. If students A and B are picked for the committee, it is the same committee if students B and A are picked.

**Method 1:** List all possibilities.

This method works best when dealing with small numbers. An easy way to come up with all possible combinations is to use a distribution method. By drawing lines from the objects in question to all other possibilities after it and repeating for all the objects, it is simple to see every combination of objects.



By counting all possibilities it can be seen that there are ten different ways to form a committee with this group of students.

**Method 2:** Use the equation for combinations.

Use the equation  ${}_nC_r = \frac{n!}{r!(n-r)!}$  where  $C$  stands for combination,  $n$  is the number of objects, and  $r$  is how many objects that are being taken at a time.

The total number of students that can be chosen from is 5, which is the  $n$  value. Because only 2 of the students can be picked for the committee, the  $r$  value is 2.

$${}_5C_2 = \frac{5!}{2!(5-2)!}$$

Simplify the denominator.

$${}_5C_2 = \frac{5!}{2!3!}$$

In the numerator  $5!$  equals 120 while in the denominator  $2!$  is multiplied with  $3!$  to equal 12.

$${}_5C_2 = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{120}{12}$$

Reduce the fraction by dividing.

$${}_5C_2 = 10$$

There are 10 ways to group these 5 different students 2 at a time.

Use TI 83/84 to calculate Combinations:

1. First input value for  $n$ .
2. Press MATH and select PRB, then choose option 3:nCr and press the ENTER key.
3. Input value for  $r$  and press the ENTER key.

**Sample Problems**

1. A quarter-operated candy machine contains candies in 5 different flavors. If the machine dispenses handfuls of 15 candies per quarter, how many possible combinations of candies could be dispensed with just one turn?
2. At a parade there are 3 marching bands, 7 floats, 2 horse-drawn carriages, and 2 teams of dancers. How many ways can these groups be arranged in the procession?
3. If you are playing with a standard 52-card deck, how many hands with 5 cards each can be drawn?
4. When creating a 4-digit PIN number for a bank account using the numbers 0-9, how many possible PIN numbers are there?
5. How many possible outcomes can be obtained when rolling two 6-sided dice?
6. In a horse race, how many possible outcomes are there for first, second, and third place if there are 27 horses racing?
7. The local animal society is putting on a pet talent show for the community. There are 10 dogs, 5 cats, 2 ferrets, and 1 gold fish entered in the talent show. How many ways can the animal society arrange the schedule for the pet talent show?
8. How many five-digit zip codes can there be, using the numbers 0-9, if it cannot begin with zero or one, the last two digits cannot be the same number, and the last digit cannot be zero?
9. On a combination lock with the numbers 0-39 on the dial, how many possible 3-number locker combinations can there be?
10. In a contest a first place prize, second place prize, and third place prize are awarded to a crowd of 60 contestants. How many outcomes are possible?
11. When playing the lottery there are 6 different balls ranging between 0 and 60 that can be chosen from. How many different possibilities are there when picking lottery numbers?
12. An art gallery is putting on an art exhibition with 3 Monet paintings, 4 Rembrandt paintings, 2 Picasso paintings, and 1 Botticelli painting. How many ways can the paintings be arranged in the art gallery?
13. There is a bag with 12 different colored marbles. If only three of the marbles are selected from the bag, how many possible color combinations can be selected?

**Answers to Sample Problems**

1. 3,003 Combination
2. 720,720 Permutation Exception
3. 2,598,960 Combination
4. 5,040 Permutation
5. 66 Combination
6. 17,550 Permutation
7. 7,351,344 Permutation Exception
8. 57,600 Fundamental Counting Principle
9. 59,280 Permutation
10. 205,320 Permutation
11. 55,525,372 Combination
12. 12,600 Permutation Exception
13. 220 Combination